ON EXTENDED STIELTJES SERIES*

BY
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1. Let

$$(1) \qquad c_0 - c_1 z + c_2 z^2 - \cdots$$

be a power series with real coefficients such that the determinants

$$A_n = \begin{vmatrix} c_0, & c_1, & \cdots, & c_{n-1} \\ c_1, & c_2, & \cdots, & c_n \\ \cdot & \cdot & \cdot & \cdot \\ c_{n-1}, & c_n, & \cdots, & c_{2n-2} \end{vmatrix}, \qquad B_n = \begin{vmatrix} c_1, & c_2, & \cdots, & c_n \\ c_2, & c_3, & \cdots, & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_n, & c_{n+1}, & \cdots, & c_{2n-1} \end{vmatrix},$$

$n=1, 2, 3, \cdots$, are all positive. Then we define a *kth extension* of (1) to be a series

$$(2) \qquad (-1)^k \frac{c_{-k}}{z^k} + (-1)^{k-1} \frac{c_{-k+1}}{z^{k-1}} + \cdots - \frac{c_{-1}}{z} + c_0 - c_1 z + c_2 z^2 - \cdots$$

such that all the determinants formed from the A_n and B_n by replacing throughout c_i by c_{i-k}, $i=0, 1, 2, 3, \cdots$, are positive.

In a previous paper† in these Transactions the present writer gave a necessary and sufficient condition for the existence of a first extension of (1), and gave examples to show that for any k there are series possessing a kth but not a $(k+1)$st extension, and others possessing extensions of infinite order. The condition there given is as follows. Let

$$(3) \qquad \frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \cdots$$

be the Stieltjes‡ continued fraction corresponding to the Stieltjes series (1). Then if $\sum a_{2i} = a_2 + a_4 + \cdots$ converges, and only then, a first extension exists and we may choose $c_{-1} \geqq \sum a_{2i}$ at pleasure. If c_{-p} exists then c_{-p-1} exists if and only if the series§ $\sum a_{2i}^{-p}$ in the continued fraction

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† H. S. Wall, *On the Padé approximants associated with the continued fraction and series of Stieltjes*, these Transactions, vol. 31 (1929), pp. 91–116, Chapter III.

‡ Stieltjes, *Recherches sur les fractions continues*, Annales de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47, 1894–95; or Oeuvres, vol. 2.

§ Here and hereafter I write the superscripts without parentheses.

$$(4)\qquad \frac{1}{a_1^{-p}} + \frac{z}{a_2^{-p}} + \frac{z}{a_3^{-p}} + \cdots$$

corresponding to the Stieltjes series

$$(5)\qquad c_{-p} - c_{-p+1}z + c_{-p+2}z^2 - \cdots$$

converges. The minimum value of c_{-p-1} is $\sum a_{2i}^{-p}$, $p=0, 1, 2, \cdots, a_n^0 \equiv a_n$.

It will be convenient to make the following definition. The kth extension of (1) in which every c_{-p}, $p=1, 2, 3, \cdots, k$, has its minimum value is the *minimal* kth extension of (1).

In the following article I shall give a necessary and sufficient condition for a minimal kth extension of (1), and then show that throughout a large class of Stieltjes series, including among others all those for which $\sum a_i = a_1 + a_2 + a_3 + \cdots$ converges,* minimal extensions of infinite order exist. Furthermore, if in this case we form the Stieltjes series

$$(6)\qquad \frac{c_{-1}}{z} - \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} - \cdots$$

with corresponding Stieltjes continued fraction

$$(7)\qquad \frac{1}{\alpha_1 z} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3 z} + \cdots$$

then the latter converges over any finite region not containing a part of the negative half of the real axis, and its limit is the limit of the even convergents of (3). The series (6) converges without a circle of known radius R to this same limit.

The next paragraph contains preliminaries.

2. In the above mentioned article I gave formulas† which may be used to connect the numbers a_i^{-p} of (4) with the a_i^{-p-1} and also with the a_i^{-p+1}. They run as follows:

$$(8)\qquad a_{2i}^{-p} = a_{2i+1}^{-p-1} \Big/ \left(\sum_{i=0}^{i-1} a_{2i+1}^{-p-1} \right) \cdot \left(\sum_{i=0}^{i} a_{2i+1}^{-p-1} \right),$$

$$(9)\qquad a_{2i-1}^{-p} = a_{2i}^{-p-1} \left(\sum_{i=0}^{i-1} a_{2i+1}^{-p-1} \right)^2,$$

* This case was treated in my article, loc. cit., p. 112, Theorem 5. The extensions there obtained were not minimal extensions.

† Wall, loc. cit., formulas (49), (50), (65), (67).

$$a_{2i}^{-p} = a_{2i-1}^{-p+1}\left(c_{-p} - \sum_{i=1}^{i-1} a_{2i}^{-p+1}\right)^2, \tag{10}$$

$$a_{2i+1}^{-p} = a_{2i}^{-p+1} \Big/ \left(c_{-p} - \sum_{i=1}^{i-1} a_{2i}^{-p+1}\right) \cdot \left(c_{-p} - \sum_{i=1}^{i} a_{2i}^{-p+1}\right). \tag{11}$$

If we solve (9) for a_{2i}^{-p-1}, replace p by $p-1$ and equate the value of a_{2i}^{-p} so found to that given by (10) we will obtain, after simple reductions,

$$c_{-p} = \sum_{i=1}^{i-1} a_{2i}^{-p+1} + 1 \Big/ \sum_{i=1}^{i} a_{2i-1}^{-p}. \tag{12}$$

Stieltjes* showed that the sequences of even and odd convergents of the continued fraction

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \cdots \tag{13}$$

always converge to limit functions $F_1(z)$ and $F_2(z)$ respectively, and that these limits are expressible as Stieltjes† integrals

$$F_1(z) = \int_0^\infty \frac{d\phi_1(u)}{z+u}, \quad F_2(z) = \int_0^\infty \frac{d\phi_2(u)}{z+u}, \tag{14}$$

where $\phi_1(u)$ and $\phi_2(u)$ are non-decreasing real functions such that $\phi_1(0) = \phi_2(0) = 0$, $\phi_1(\infty) = \phi_2(\infty) = 1/a_1$. The formal expansion of either integral into a power series $P(1/z)$ gives the Stieltjes series corresponding to (13), namely

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots, \tag{15}$$

and accordingly $\phi_1(u)$ and $\phi_2(u)$ are functions $\phi(u)$ satisfying the equations

$$\int_0^\infty u^i d\phi(u) = c_i \qquad (i = 0, 1, 2, \cdots). \tag{16}$$

When $\sum a_i$ diverges, $F_1(z) \equiv F_2(z)$, and all functions $\phi(u)$ satisfying (16) are *equivalent*, i.e. equal at all points of continuity. On the other hand, when $\sum a_i$ converges, $F_1(z) \not\equiv F_2(z)$ and there is an infinite number of non-equivalent functions $\phi(u)$ satisfying (16). In this case the integrals (14) reduce to infinite series of the form

* Stieltjes, loc. cit., §§47–48. Note that (13) becomes (3) if we replace z by $1/z$ and then drop the factor z.

† Stieltjes, loc. cit., §38. Cf. also O. Perron, *Die Lehre von den Kettenbrüchen*, 1913, Chapter IX, for the definition and essential properties of Stieltjes integrals, and the chief results of Stieltjes.

$$(17)\qquad F_1(z) = \sum_{i=1}^{\infty} \frac{\mu_i}{z+\lambda_i}, \quad F_2(z) = \frac{\nu_0}{z} + \sum_{i=1}^{\infty} \frac{\nu_i}{z+\theta_i}$$

in which μ_i, λ_i, ν_i, θ_i are all real and positive; and (16) for $\phi(u)=\phi_1(u)$ become

$$(18)\qquad \sum_{i=1}^{\infty} \lambda_i^p \mu_i = c_p \qquad (p = 0,1,2,\cdots),$$

with similar equations for $\phi=\phi_2$.

3. These preliminary remarks having been made, I shall prove the following theorem.

THEOREM 1. *The Stieltjes series* (1) *admits a first extension when and only when the integral*

$$(19)\qquad \int_0^{\infty} \frac{d\phi_1(u)}{u}$$

converges. When this condition is fulfilled we may choose c_{-1} equal to (19) *or any greater number.*

For the proof of this theorem the following lemmas will be needed.

LEMMA 1. *If the Stieltjes integrals*

$$\int_0^{\infty} u^k d\phi(u) = c_k \quad (k = 0,1,2,\cdots), \quad \textit{and} \quad \phi_1(u) = \int_0^u \frac{d\phi(u)}{u^n},$$

where u is real and positive and n is a positive integer, exist, then $\phi_1(u)$, which is real, non-negative, and non-decreasing, satisfies the equations

$$\int_0^{\infty} u^{n+k} d\phi_1(u) = c_k \qquad (k = 0,1,2,3,\cdots).$$

LEMMA 2. *If*

$$\phi_1(u) = \int_0^u u^n d\phi(u),$$

where n is a positive or negative integer or 0, *and $\phi(u)$ satisfies the equations*

$$\int_0^{\infty} u^{n+k} d\phi(u) = c_k \qquad (k = 0,1,2,3,\cdots),$$

is convergent, then

$$\int_0^{\infty} u^k d\phi_1(u) = c_k \qquad (k = 0,1,2,3,\cdots).$$

According to the definition of a Stieltjes integral, divide the interval $(0, b)$, $b>0$, in m sub-intervals by the points $(x_0=0<x_1<x_2<\cdots<x_m=b)$, and let the norm of the division be δ. Then if $x_{i-1}\leqq\xi_i\leqq x_i$,

$$\begin{aligned}\int_0^b u^{n+k}d\phi_1(u) &= \lim_{\delta=0}\sum_{i=1}^m \xi_i^{n+k}\left[\int_0^{x_i}\frac{d\phi(u)}{u^n}-\int_0^{x_{i-1}}\frac{d\phi(u)}{u^n}\right]\\ &= \lim_{\delta=0}\sum_{i=1}^m \xi_i^{n+k}\int_{x_{i-1}}^{x_i}\frac{d\phi(u)}{u^n}\\ &= \lim_{\delta=0}\sum_{i=1}^m \xi_i^{n+k}\frac{1}{\xi_i'^{\,n}}[\phi(x_i)-\phi(x_{i-1})],\end{aligned}$$

where ξ_i' is a properly chosen point between x_{i-1} and x_i.* But since $\phi_1(u)$ is a non-decreasing, non-negative, real function, and u^{n+k} is continuous in the interval $(0, b)$, the integral $\int_0^b u^{n+k}d\phi_1(u)$ exists. Consequently we may take $\xi_i=\xi_i'$ and the above limit becomes

$$\int_0^b u^{n+k}d\phi_1(u) = \int_0^b u^k d\phi(u) \qquad (k=0,1,2,\cdots).$$

Now the integral on the right has a limit for $b=\infty$. Hence the integral on the left has a limit for $b=\infty$ and these limits are equal. This proves Lemma 1.

To prove the second lemma, we choose b and $x_0, x_1, x_2, \cdots, x_m$ as above and form the sum

$$\sum_{i=1}^m \xi_i^k\left[\int_0^{x_i}u^n d\phi(u)-\int_0^{x_{i-1}}u^n d\phi(u)\right] = \sum_{i=1}^m \xi_i^k\int_{x_{i-1}}^{x_i}u^n d\phi(u), \tag{20}$$

which is equal to

$$\sum_{i=1}^m \xi_i^k\xi_i'^{\,n}[\phi(x_i)-\phi(x_{i-1})],$$

where ξ_i' is a properly chosen point between x_{i-1} and x_i. But since ξ_i is an arbitrary point in this interval we may take $\xi_i=\xi_i'$. Hence the last sum is equal to

$$\sum_{i=1}^m \xi_i^{n+k}[\phi(x_i)-\phi(x_{i-1})]$$

* The theorem here used, which corresponds to the mean value theorem for Riemannian integrals and is proved similarly, is as follows. If $f(x)$ is continuous for $a\leqq x\leqq b$, and $\phi(x)$ is non-decreasing and non-negative then there exists some point ξ, $a\leqq\xi\leqq b$, such that

$$\int_a^b f(x)d\phi(x) = f(\xi)[\phi(b)-\phi(a)].$$

If $f(x)$ is continuous only for $a<x\leqq b$, and $\lim_{x=a+}f(x)=+\infty$, the same equation holds with $a<\xi\leqq b$.

which by hypothesis has the limit c_k for $\delta=0$, $b=\infty$. Consequently the left member of (20) has the limit c_k for $\delta=0$, $b=\infty$, and this limit is the integral $\int_0^\infty u^k d\phi_1(u)$. This proves Lemma 2.

We now prove that the condition of Theorem 1 is sufficient for a first extension of (1). Assume that (15) converges and set

$$\phi^{-1}(u) = \int_0^u \frac{d\phi_1(u)}{u}, \qquad \phi^{-1}(0) = 0.$$

Since $\phi_1(u)$ is a solution of (16) we have, by Lemma 1 with $n=1$,

$$\int_0^\infty u^{1+i} d\phi^{-1}(u) = c_i \qquad (i = 0,1,2,\cdots).$$

Thus if

$$\int_0^\infty d\phi^{-1}(u) = \int_0^\infty \frac{d\phi_1(u)}{u} = c_0' \ ; \ c_{i-1} = c_i' \qquad (i = 1,2,3,\cdots),$$

the following equations hold:

$$\int_0^\infty u^i d\phi^{-1}(u) = c_i' \qquad (i = 0,1,2,\cdots).$$

It then follows from the work of Stieltjes that c_0', c_0, c_1, $\cdots$ are coefficients in a Stieltjes series. The sufficiency of the condition is thus proved.

To prove the necessity of the condition, assume that a first extension of (1) exists, and consider separately the cases $\sum a_i$ diverges, $\sum a_i$ converges, respectively.

(a) If $\sum a_i$ diverges, then $c_{-1}=\sum a_{2i}+\delta$, where $\delta \geqq 0$ (§1). If $\delta=0$ it follows from (12) with $p=1$, that $\sum a_{2i-1}^{-1}$ must diverge; and if $\delta>0$, we see from (10) with $p=1$ that $\sum a_{2i}^{-1}$ diverges. Hence in either case $\sum a_i^{-1}$ diverges, and consequently the continued fraction (4) with $p=1$ converges to the limit

$$\frac{1}{z}\int_0^\infty \frac{d\phi^{-1}(u)}{z^{-1}+u}$$

and

$$\int_0^\infty u^{1+i} d\phi^{-1}(u) = c_i \qquad (i = 0,1,2,\cdots).$$

Therefore by Lemma 2 with $n=1$, $\phi(u)=\phi^{-1}(u)$, the function

$$\psi_1(u) = \int_0^u u\, d\phi^{-1}(u)$$

is a solution of (16), and since $\sum a_i$ diverges this function is equivalent to $\phi_1(u)$.

Let now a, b be real and positive and points of continuity* of $\phi_1(u)$. Then if $b>a$ it follows that

$$\int_a^b \frac{d\phi_1(u)}{u} = \int_a^b \frac{d\psi_1(u)}{u} = \lim_{\delta=0} \sum_{i=1}^{m} \frac{1}{\xi_i'} \cdot \xi_i' \left[\phi^{-1}(x_i) - \phi^{-1}(x_{i-1})\right],$$

where ξ_i' is a properly chosen point between x_{i-1} and x_i, $i=1, 2, \cdots, m$, $x_0=a$, $x_m=b$. Thus if $b'>b$,

$$\int_a^{b'} \frac{d\phi_1(u)}{u} = \int_a^b d\phi^{-1}(u) + \int_b^{b'} \frac{d\phi_1(u)}{u}.$$

Now since $\int_0^\infty d\phi_1(u)$ converges, $\int_b^\infty d\phi_1(u)/u$ will surely converge if $b \geqq 1$. Hence for any $\epsilon>0$, there exists a number B such that if $b>B$, $b'>b$,

$$\left| \int_b^{b'} \frac{d\phi_1(u)}{u} \right| < \epsilon,$$

and consequently

$$\lim_{b'=\infty} \int_a^{b'} \frac{d\phi_1(u)}{u} = \int_a^\infty d\phi^{-1}(u) = \phi^{-1}(\infty) - \phi^{-1}(a),$$

or

$$\int_a^\infty \frac{d\phi_1(u)}{u} = c_{-1} - \phi^{-1}(a). \tag{21}$$

If now a approaches 0, over points of continuity of $\phi_1(u)$, the left member of (21) will have the limit†

$$\lim_{a=0^+} \int_a^\infty \frac{d\phi_1(u)}{u} = c_{-1} - \frac{1}{\sum a_{2i-1}^{-1}}. \tag{22}$$

Let a_1 be another point of continuity of $\phi_1(u)$ and let $0<a_1<a'<a$. Then

$$\int_{a'}^\infty \frac{d\phi_1(u)}{u} = \int_{a_1}^\infty \frac{d\phi_1(u)}{u} - \int_{a_1}^{a'} \frac{d\phi_1(u)}{u},$$

or simply

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* Note that $\phi_1(u)$, being monotone, has points of continuity everywhere dense in the interval $(0, \infty)$.

† Cf. Stieltjes, loc. cit., §58.

(23) $$\int_{a'}^{\infty} = \int_{a_1}^{\infty} - \int_{a_1}^{a'}.$$

Now

$$\int_{a_1}^{a'} = \frac{1}{\xi}\left[\phi_1(a') - \phi_1(a_1)\right], \quad a_1 \leqq \xi \leqq a',$$

and since $\phi_1(u)$ is continuous at a_1 we may make

(24) $$\left| \int_{a_1}^{a'} \right| < \frac{\epsilon}{2}, \quad \text{if } \epsilon > 0,\ a' - a_1 < \delta.$$

Then by (22), (23), (24),

$$\int_{a'}^{\infty} = c_{-1} - 1\Big/ \sum a_{2i-1}^{-1} + \epsilon, \text{ if } a_1 < \eta,\ a' - a_1 < \delta.$$

Consequently

$$\lim_{a' = 0^+} \int_{a'}^{\infty} \frac{d\phi_1(u)}{u} = c_{-1} - 1\Big/ \sum a_{2i-1}^{-1}.$$

But by (12) with $p = 1$, $c_{-1} = \sum a_{2i} + 1/\sum a_{2i-1}^{-1}$, and therefore

$$\lim_{a' = 0^+} \int_{a'}^{\infty} \frac{d\phi_1(u)}{u} = \int_{0}^{\infty} \frac{d\phi_1(u)}{u} = \sum a_{2i} \leqq c_{-1}.$$

This completes the proof of the theorem for the case that $\sum a_i$ is divergent.

(b) When $\sum a_i$ converges, $\int_0^\infty d\phi_1(u)/(z+u)$ reduces to the first series (17) and therefore if $0 < a < \lambda_1$, supposing $\lambda_1 < \lambda_2 < \cdots$, this integral is equal to $\int_a^\infty d\phi_1(u)/(z+u)$. It then follows by a known theorem* that this integral represents an analytic function for any z not contained in the interval $(-\infty, -a)$. Consequently $\int_0^\infty d\phi_1(u)/u$ converges. Furthermore,

$$\int_{0}^{\infty} \frac{d\phi_1(u)}{u} = \sum_{i=1}^{\infty} \frac{\mu_i}{\lambda_i} = \lim_{n=\infty} \frac{P_{2n}(0)}{Q_{2n}(0)} = \sum a_{2i} \leqq c_{-1}$$

inasmuch as $P_{2n}(z)/Q_{2n}(z)$, the $2n$th convergent of (13), has the value $\sum_{i=1}^{n} a_{2i}$ when $z = 0$. This completes the proof of Theorem 1.

THEOREM 2. *The Stieltjes series* (1) *admits a minimal kth extension when and only when the integral*

(25) $$\int_{0}^{\infty} \frac{d\phi_1(u)}{u^k}$$

converges.

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* Perron, loc. cit., p. 369.

Suppose first that (25) converges. Then $\int_0^\infty d\phi_1(u)/u^p$, $p<k$, converges. For if $0<x<x'<\delta<1$,

$$\int_x^{x'} \frac{d\phi_1(u)}{u^p} < \int_x^{x'} \frac{d\phi_1(u)}{u^k} < \epsilon,$$

if δ is sufficiently small.

Taking $p=1$ it follows from Theorem 1 that a first extension exists, and if

$$c_{-1} = \int_0^\infty d\phi_1(u)/u = \sum a_{2i},$$

$\sum a_i^{-1}$ must diverge by (12). Consequently

$$\phi^{-1}(u) = \int_0^u d\phi_1(u)/u.$$

Then taking $p=2$ we find that

$$\int_0^\infty \frac{d\phi^{-1}(u)}{u} = \int_0^\infty \frac{d\phi_1(u)}{u^2} \tag{26}$$

converges and again by Theorem 1, a second extension exists and we take c_{-2} equal to (26), etc. Continuing this argument one will finally arrive at a minimal kth extension of (1).

On the other hand suppose that (1) admits a minimal kth extension, $k\geqq 1$. Then by Theorem 1,

$$\int_0^\infty d\phi_1(u)/u = \sum a_{2i}$$

converges, and c_{-1} has this value. Then by (12) $\sum a_i^{-1}$ diverges and therefore

$$\phi^{-1}(u) = \int_0^u d\phi_1(u)/u.$$

If $k\geqq 2$ it follows from Theorem 1 that

$$\int_0^\infty d\phi^{-1}(u)/u = \int_0^\infty d\phi_1(u)/u^2 = \sum a_{2i}^{-1}$$

converges and is equal to c_{-2}. Hence

$$\phi^{-2}(u) = \int_0^u d\phi_1(u)/u^2,$$

and if $k\geqq 3$,

$$\int_0^\infty d\phi_1(u)/u^3 = \sum a_{2i}^{-2}$$

converges, etc. This argument may evidently be continued until we arrive at the integral $\int_0^\infty d\phi_1(u)/u^k$, whatever value k may have.

4. We next prove the theorem mentioned at the end of §1, namely

THEOREM 3. (a) *If there exist a number $a>0$ such that*

$$\int_0^\infty d\phi_1(u) = \int_a^\infty d\phi_1(u), \tag{27}$$

then (1) *admits a minimal kth extension for all values of k.*

(b) *The continued fraction* (7) *converges to the limit*

$$F_1(z) = \int_0^{1/a} \frac{-ud\phi_1(1/u)}{z+u} \tag{28}$$

which is the limit of the even convergents of (3).

(c) *The series* (6) *converges for all z for which* $|z|>1/a$, *and represents* $F_1(z)$ *in that region.*

(d) *In case* $\sum a_i$ *converges, a may be chosen arbitrarily in the open interval* $(0, \lambda_1)$, *and* $c_{-p}=\sum_{i=1}^\infty \mu_i/\lambda_i^p$, $p=1, 2, 3, \cdots$.

For by (27)

$$\int_0^\infty d\phi_1(u)/u^k = \int_a^\infty d\phi_1(u)/u^k,$$

and this integral is readily seen to be convergent. Hence, by Theorem 2, (1) admits a minimal kth extension. Consider now the integral (28). We have

$$F_1(z) = \int_0^{1/a} \frac{-ud\phi_1(1/u)}{z+u} = \int_0^{1/a} -u\left[\frac{1}{z} - \frac{u}{z^2} + \frac{u^2}{z^3} - \cdots\right] d\phi_1(1/u).$$

Since the series within the brackets converges uniformly over $(0, 1/a)$ if $|z|>\delta>1/a$, it may be integrated term by term. Therefore

$$\begin{aligned} F_1(z) &= \frac{-\int_0^{1/a} ud\phi_1(1/u)}{z} + \frac{\int_0^{1/a} u^2 d\phi_1(1/u)}{z^2} - \cdots \\ &= \frac{\int_a^\infty d\phi_1(u)/u}{z} - \frac{\int_a^\infty d\phi_1(u)/u^2}{z^2} + \cdots \\ &= \frac{c_{-1}}{z} - \frac{c_{-2}}{z^2} + \cdots, \end{aligned}$$

convergent if $|z|>1/a$. It follows* that the continued fraction (7) converges and is equal to $F_1(z)$. When $\sum a_i$ converges the integrals $\int_0^\infty d\phi_1(u)/u^k$ evidently reduce to the sums $\sum_{i=1}^\infty \mu_i/\lambda_i^k$ by (17).

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* Cf. Stieltjes, loc. cit., §10, in which it is shown that when a Stieltjes series converges, the numbers $1/(\alpha_i\alpha_{i+1})$, $i=1, 2, 3, \cdots$, must increase to a finite limit, and consequently $\sum \alpha_i$ must diverge, thus implying the convergence of the continued fraction.

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